2. ST $\int_0^{\pi/2} \sin^p x\,dx \int_0^{\pi/2} \sin^{p+1} x\,dx = \frac{\pi}{2(p+1)}$

$$I = \int_0^{\pi/2} \sin^p x \cdot \cos^0 x\,dx \int_0^{\pi/2} \sin^{p+1} x \cos^0 x\,dx$$

$$= \frac{1}{2} B\left(\frac{p+1}{2}, \frac{0+1}{2}\right) \cdot \frac{1}{2} B\left(\frac{p+1+1}{2}, \frac{0+1}{2}\right)$$

$$= \frac{1}{4} \frac{\Gamma\frac{p+1}{2}\,\Gamma\frac{1}{2}}{\cancel{\Gamma\left(\frac{p+1}{2}+\frac{1}{2}\right)}} \cdot \frac{\cancel{\Gamma\frac{p+2}{2}}\,\Gamma\frac{1}{2}}{\Gamma\left(\frac{p+2}{2}+\frac{1}{2}\right)}$$

$$= \frac{1}{4} \sqrt{\pi}\sqrt{\pi} \cdot \frac{\Gamma\frac{p+1}{2}}{\Gamma\frac{p+3}{2}}$$

$$= \frac{\pi}{\cancel{4}_{2}} \frac{\cancel{\Gamma\frac{p+1}{2}}}{\frac{p+1}{\cancel{2}}\,\cancel{\Gamma\frac{p+1}{2}}} = \frac{\pi}{2(p+1)}$$

Q5 S.T. $\int_{-1}^{1}(1+x)^m(1-x)^n\,dx = 2^{m+n+1}\beta(m+1,n+1)$.

Let $1+x=2t$ $\therefore dx=2dt$

$\therefore 1-x=2-2t$

x	-1	1
t	0	1

$$I=\int_0^1 (2t)^m(2-2t)^n\,2dt$$

$$=\int_0^1 2^m \cdot 2^n \cdot 2\,(t)^m(1-t)^n\,dt$$

$$=2^{m+n+1}\beta(m+1,n+1)$$

Q6 If $\beta(n,3)=\frac{1}{3}$, find n.

$$\frac{\Gamma n\,\Gamma 3}{\Gamma n+3}=\frac{1}{3}$$

$$\therefore \frac{\Gamma n \cdot 2\times 1}{(n+2)(n+1)(n)\Gamma n}=\frac{1}{3}$$

$\therefore n(n+1)(n+2) = 6$

$n(n^2+3n+2) = 6$

$n^3+3n^2+2n-6 = 0$

If $n = 1$ then LHS = RHS

$\therefore n = 1$

Q7 S.T. $\displaystyle\int_{-1}^{1} \frac{(1+x)^{2m-1}(1-x)^{2n-1}}{(1+x^2)^{m+n}}\,dx = 2^{m+n-2}\beta(m,n)$

Let $x = \dfrac{y-1}{y+1}$ $\therefore dx = -\dfrac{(y-1)}{(y+1)^2} + \dfrac{1}{y+1}\,dy$

$\therefore y-1 = xy + x$ $\qquad dx = \dfrac{-y+1+y+1}{(y+1)^2} = \dfrac{2}{(y+1)^2}\,dy$

$y - xy = x+1$

$\therefore y = \dfrac{1+x}{1-x}$

x	-1	1
y	0	∞

$$I = \int_0^\infty \left(\frac{y+1+y-1}{y+1}\right)^{2m-1} \cdot \left(\frac{y+1-y+1}{y+1}\right)^{2n-1} \Bigg/ \left(\frac{y^2+2y+1+y^2-2y+1}{y^2+2y+1}\right)^{m+n} \cdot \frac{2dy}{(y+1)^2}$$

$$I = \int_0^\infty \frac{\left(\frac{2y}{y+1}\right)^{2m-1} \left(\frac{2}{y+1}\right)^{2n-1}}{\left(\frac{2y^2+2}{(y+1)^2}\right)^{m+n}} \cdot \frac{2}{(y+1)^2}\, dy$$

$$= \int_0^\infty \frac{2^{2m-1+2n-1-m-n+1} \cdot y^{2m-1} \cdot (y^2+1)^{-m-n}}{(y+1)^{2m-1+2n-1-2m-2n+2}}\, dy$$

$$2^{m+n-1} \int_0^\infty \frac{y^{2m-1}}{(1+y^2)^{m+n}}\, dy \cdot \frac{1}{(y+1)^0}$$

Let $y^2 = t$ $\therefore y = t^{1/2}$ $\therefore dy = \frac{1}{2} t^{-1/2} dt$

$$= 2^{m+n-1} \int_0^\infty \frac{t^{m-\frac{1}{2}}}{(1+t)^{m+n}} \cdot \frac{1}{2} t^{-1/2}\, dt$$

$$= 2^{m+n-2} \int_0^\infty \frac{t^{m-1}}{(1+t)^{m+n}}\, dt$$

$$= 2^{m+n-2} B(m,n)$$

Q8 ST $\beta(m,m)\,\beta\left(m+\frac{1}{2}, m+\frac{1}{2}\right)$

$$= \frac{\pi}{m} 2^{1-4m}.$$

$$LHS = \frac{\Gamma m \, \Gamma m}{\Gamma 2m} \cdot \frac{\Gamma m+\frac{1}{2} \cdot \Gamma m+\frac{1}{2}}{\Gamma 2m+1}$$

$$= \Gamma m \, \Gamma m+\tfrac{1}{2} \cdot \Gamma m \cdot \Gamma m+\tfrac{1}{2} \cdot \frac{1}{\Gamma 2m \cdot 2m \Gamma 2m}$$

$$= \left(\frac{\Gamma m \, \Gamma m+\frac{1}{2}}{\Gamma 2m}\right)^2 \cdot \frac{1}{2m}$$

$$= \left(\frac{\sqrt{\pi}}{2^{2m-1}}\right)^2 \cdot \frac{1}{2m}$$

$$= \frac{\pi}{2^{4m-2}} \cdot \frac{1}{2m}$$

$$= \frac{\pi}{m} \cdot 2^{2-4m-1}$$

$$= \frac{\pi}{m} 2^{1-4m}$$

Q9 Establish Red. formula of β funct

i.e. P.T. $\beta(m,n) = \frac{(m-1)(n-1)}{(m+n-1)(m+n-2)} \beta(m-1, n-1)$

$LHS = \frac{\Gamma m \, \Gamma n}{\Gamma m+n} = \frac{(m-1)\Gamma m-1 \, (n-1)\Gamma n-1}{(m+n-1)(m+n-2)\Gamma m+n-2}$

$= \frac{(m-1)(n-1)}{(m+n-1)(m+n-2)} \frac{\Gamma m-1 \, \Gamma n-1}{\Gamma m+n-2}$

$= \frac{(m-1)(n-1)}{(m+n-1)(m+n-2)} \beta(m-1, n-1)$

Q10 Solve $\int_{-\pi/4}^{\pi/4} (\cos\theta + \sin\theta)^{1/3} d\theta$ in terms of β funct.

$I = (\sqrt{2})^{1/3} \int_{-\pi/4}^{\pi/4} \left(\frac{1}{\sqrt{2}} \cos\theta + \frac{1}{\sqrt{2}} \sin\theta\right)^{1/3} d\theta$

$$I = 2^{1/6} \int_{-\pi/4}^{\pi/4} \sin^{1/3}\left(\theta + \frac{\pi}{4}\right) d\theta$$

Let $\theta + \frac{\pi}{4} = t$ $\quad \therefore d\theta = dt$

θ	$-\pi/4$	$\pi/4$
t	0	$\pi/2$

$$I = 2^{1/6} \int_{0}^{\pi/2} \sin^{1/3} t \cdot \cos^{0} t \, dt$$

$$= 2^{1/6} \cdot \frac{1}{2} \beta\left(\frac{\frac{1}{3}+1}{2}, \frac{0+1}{2}\right)$$

$$= 2^{1/6-1} \beta\left(\cancel{\frac{4}{6}}\frac{2}{3}, \frac{1}{2}\right)$$

Q12 S7. $1 \cdot 3 \cdot 5 \cdots (2n-1) = \dfrac{2^n \, \Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}}$

$$\Gamma\left(n+\frac{1}{2}\right) = \left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)\cdots\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\,\Gamma\left(\frac{1}{2}\right)$$

$$= \frac{(2n-1)(2n-3)\cdots 5\cdot 3\cdot 1}{2^n}\cdot\sqrt{\pi}$$

$$\therefore \frac{2^n}{\sqrt{\pi}}\,\Gamma\left(n+\frac{1}{2}\right) = 1\cdot 3\cdot 5\cdots(2n-1)$$